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Abstract Wave Equations with a Singular Nonlinear Forcing Term

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Let $V \subset H$ be real separable Hilbert spaces. The abstract wave equation $u'' + A(t)\,u = g(u)$, where $u(t) \in V$, $A(t)$ maps V to its dual V^*, and g is a nonlinear map from the ball $S(R_0) = \{u \in V: \|u\| < R_0\}$ into H, is considered. It is assumed that g is locally Lipschitz in $S(R_0)$ and possibly singular at the boundary. Local existence and continuation theorems are established for the Cauchy problem $u(0) = u_0 \in S(R_0)$, $u'(0) = u_1 \in H$. Global existence is shown for $g(u) = \varepsilon\phi(u)$, where ϕ has a potential and ε is small. Global nonexistence is shown for $g(u) = \varepsilon\phi(u)$, where ϕ satisfies an abstract convexity property and ε is large.

0. INTRODUCTION

Recently [2] the nonlinear initial boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= \varepsilon\phi(u), && 0 < x < 1, t > 0, \\ u(t, 0) = u(t, 1) &= 0, && t > 0, \\ u(0, x) = u_t(0, x) &= 0, && 0 < x < 1, \end{aligned} \tag{0.1}$$

where $\phi \in C^1(-\infty, M)$, $M > 0$, is a convex function with $\lim_{u \to M^-} \phi(u) = +\infty$, was shown to possess global solutions (in time) provided that ε is small. It was also shown that for ε large all solutions of (0.1) must quench in finite time. By quenching we mean that the solution $u(t, x)$ reaches M in finite time and thus $\phi(u)$ becomes infinite. Changing variables in (0.1) $\bar{x} = xL$, $\bar{t} = tL$, and setting $\varepsilon = L^2$ results in (0.1) again with ε replaced by 1 and x varying between 0 and L. In this way we can interpret the stated results as saying that global solutions exist for short semilinear strings but do not exist for long semilinear strings. This paper is a result of our investigation of this phenomena in several dimensions.

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We consider an abstract version of (0.1)

$$\frac{d^2u}{dt^2} + A(t)\,u = g(u), \qquad 0 < t < +\infty,$$
$$u(0) = u_0 \in V, \qquad u'(0) = u_1 \in H, \tag{0.2}$$

where $V \subset H$ are real separable Hilbert spaces, $A(t)$ is a linear operator from V to its dual V^* of elliptic type, and g is a nonlinear map from a subset of V into H. The equations (0.2) are understood as equations in Hilbert space. This includes (0.1) if we take $V = H_0^1(0, 1)$, $H = L^2(0, 1)$, and $A = [-\partial^2/\partial x^2]$.

Since we have in mind $g(u) = \varepsilon\phi(u)$ as an application we must deal with the possibility that g may not be defined on all of V nor on all of the domain of $A(t)$, $\{u \in V: A(t)u \in H\}$, for each t. This is a situation that has not been considered in the literature to our knowledge. For instance in Reed [6] the nonlinearity g must be defined at least on the domain of A (independent of t); no such requirement is made here. Instead we assume that g is defined on some ball about the origin in V. In the applications this apparently requires $V \subset L^\infty(\Omega)$, where $H = L^2(\Omega)$. Although we wanted to apply our work to the n-dimensional wave equation, the fact that $H^1(\Omega) \not\subset L^\infty(\Omega)$ (cf. Adams [1]) for $n \geqslant 2$ excluded this possibility. Of course if g is globally defined (for example if $g \in C(-\infty, +\infty)$ and g is Lipschitz) then our local existence theorem (Sect. 2) can be applied to the n-dimensional wave equation yielding a well-known result.

Our analysis differs from that in [6] in another important way. The topology we use is that of continuous functions from the real line (or an interval) into the Hilbert space V. The norm of the derivative, $u'(t)$, is not used. In this way we are able to show local existence provided that u_0 is in a certain ball; there is no constraint on u_1. This approach was necessary for the quenching theory in Section 4 and provided a more natural way for dealing with the type of singular forcing term considered. However, using this approach we cannot get an estimate on the interval of existence which is uniform over the ball containing u_0.

In Section 1 we collect some facts regarding the associated linear problem from the work of Lions and Magenes [5]. In Section 2 we prove a local existence and continuation theorem for (0.2) in which g is not necessarily globally defined nor defined on the domain of $A(t)$. Then in Sections 3 and 4 we take $g(u) = \varepsilon\phi(u)$ where $\varepsilon > 0$; we prove a global existence theorem for ε sufficiently small and show that global solutions cannot exist for ε large provided that ϕ satisfies an abstract convexity property. In the applications it turns out that we need an eigenfunction for A which is positive on Ω in order that a real convex function ϕ satisfy this abstract convexity property.

Our results can be applied to (0.1) to reprove the results stated above and extend them to include nonzero initial data. In Section 5, however, we consider, as a simple illustration, the problem

$$\begin{gathered} u_{tt} + \Delta^2 u = \varepsilon\phi(u), \qquad (t,x) \in \mathbb{R}^+ \times \Omega, \\ u(t,x) = \Delta u(t,x) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \partial\Omega, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \qquad x \in \Omega, \end{gathered} \tag{0.3}$$

where $\Omega = (0, L) \times (0, 1) \subset \mathbb{R}^2$, $0 < L \leqslant 1$, the function ϕ has the same properties as in (0.1), $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, and $u_1 \in L^2(\Omega)$. We show that global solutions exist for small ε but fail to exist for large ε.

1. The Linear Problem

Let $V \subset H$ be Hilbert spaces, the containment being continuous and injective. We assume V is dense in H and use $((\cdot, \cdot))$, $(\cdot, \cdot)$ and $\|\cdot\|$, $|\cdot|$ to denote the inner products and norms respectively. Let $a(t; u, v)$ be a family of bilinear forms on $V \times V$ satisfying $a(\cdot; u, v) \in C^1(\mathbb{R}^+)$ for all u, $v \in V$ ($\mathbb{R}^+ = [0, +\infty)$); $a(t; u, v) = a(t; v, u)$ for all u, $v \in V$, $t \in \mathbb{R}^+$; and $a(t; u, u) + \lambda_0 |u|^2 \geqslant \alpha_0 \|u\|^2$ for all $u \in V$, for some $\lambda_0, \alpha_0 \in \mathbb{R}^+$, $\alpha_0 > 0$. We assume there is a constant $c > 0$ such that $|a'(t; u, v)| \leqslant c \|u\| \|v\|$ for $t \in R^+$ and all u, $v \in V$ where $a'(t; u, v)$ denotes the derivative of $a(t; u, v)$ with u, $v \in V$ held fixed. Let $A(t)$: $V \to V^*$, the dual of V, be defined by $(A(t)\, u, v) = a(t; u, v)$. Let $0 < T < +\infty$ and suppose $u_0 \in V$, $u_1 \in H$, and $f \in L^2(0, T; H)$. We record the following results from Lions and Magenes [5, ch. III, Sect. 8].

THEOREM 1.1. *The (hyperbolic) initial value problem*

$$\frac{d^2u}{dt^2} + A(t)\, u = f, \qquad 0 < t < T, \tag{1.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{1.2}$$

has a unique (weak) solution $u \in B(T) = \{u \in C([0, T];\ V) | u' = du/dt \in C([0, T]; H)\}$. *In addition we have energy equality for all* $t \in [0, T]$; *that is*

$$a(t; u(t), u(t)) + |u'(t)|^2 = a(0; u_0, u_0) + |u_1|^2 + \int_0^t \{a'(s; u(s), u(s)) + 2(f(s), u'(s))\}\, ds. \tag{1.3}$$

We make a few remarks regarding the weak problem above. By a solution of (1.1) and (1.2) on $(0, T)$ we mean that $u(0) = u_0$ in H and for all $\phi \in \Phi(T) = \{\phi \in C^\infty[0, +\infty) : \phi(T) = \phi'(T) = 0\}$, $v \in V$ we have

$$\int_0^T [-(u', \phi' v) + a(t; u, \phi v)]\, dt = \int_0^T (f, \phi v)\, dt + (u_1, \phi(0)\, v).$$

Equivalently we can replace ϕv by $\psi \in \Psi(T) = \{\psi \in L^2(0, T; V)$: $\psi' \in L^2(0, T; H)$ and $\psi(T) = 0\}$. Since $C_0^\infty(0, T) \subset \Phi(T)$ we see that $(u', v)' + a(t; u, v) = (f, v)$ in the sense of distributions on $(0, T)$ for every $v \in V$. We can therefore assert that $(u', v)' + a(t; u, v) = (f, v)$, (a.e.) $t \in (0, T)$, for each $v \in V$. Moreover, if f is continuous then $(u', v)'$ exists in the classical sense (cf. Hormander [4, p. 10]) and $(u', v)' + a(t; u, v) = (f, v)$ for all $t \in (0, T)$.

Next we record some further observations on the linear problem in the form of a few lemmas. We define a linear map $K: V \times H \times L^2(0, T; H) \to B(T)$ by the unique correspondence $(u_0, u_1, f) \to u$ solving (1.1) and (1.2).

LEMMA 1.2. *Let $f \in L^2(0, T; H)$ and let $u = K(u_0, u_1, f)$; that is, u is the unique solution of* (1.1) *and* (1.2). *If $u|_{(0,t)}$ and $f|_{(0,t)}$ denote the restrictions of u and f respectively to the interval $(0, t) \subset (0, T)$ then $u|_{(0,t)} = K(u_0, u_1, f|_{(0,t)})$; that is, $u|_{(0,t)}$ is the unique solution of*

$$u'' + A(s)\, u = f|_{(0,t)}, \qquad 0 < s < t, \tag{1.4}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{1.5}$$

Proof. Since u satisfies (1.1) and (1.2) we have

$$\int_0^T [-(u', \psi') + a(t; u, \psi)]\, dt = \int_0^T (f, \psi)\, dt + (u_1, \psi(0))$$

for all $\psi \in \Psi(T)$. But by zero extension $\Psi(t) \subset \Psi(T)$ for $0 < t < T$. Thus $u|_{(0,t)}$ solves (1.4) and (1.5) and by uniqueness $u|_{(0,t)} = K(u_0, u_1, f|_{(0,t)})$.

LEMMA 1.3. *Suppose that z satisfies*

$$z'' + A(t)\, z = f, \qquad 0 < t < T + \tau$$

$$z(0) = 0, \qquad z'(0) = 0,$$

where $T > 0$, $\tau > 0$, $f \in L^2(0, T + \tau; H)$ and $f(t) = 0$ for $0 < t < T$. Then there exists a function $c(\tau) = c_1 e^{c_2 \tau}$, for constants $c_1 \geqslant 1$, $c_2 \geqslant 0$, such that

$$\{\|z(t)\|^2 + |z'(t)|^2\}^{1/2} \leqslant c(\tau) \left\{ \int_T^t |f(s)|^2\, ds \right\}^{1/2}, \quad T \leqslant t \leqslant T + \tau.$$

Proof. By the previous lemma $z(t)=0$, $0\leqslant t\leqslant T$. From the hypotheses on $a(t;u,v)$ and the energy equality we obtain, for some $c>0$,

$$\alpha_0\|z(t)\|^2+|z'(t)|^2$$
$$\leqslant\lambda_0|z(t)|^2+c\int_T^t\|z(s)\|^2\,ds+\int_T^t(|f(s)|^2+|z'(s)|^2)\,ds.$$

Writing $z(t)$ as the integral of $z'(s)$ from T to t we find, for some $c>0$,

$$\min(1,\alpha_0)\{\|z(t)\|^2+|z'(t)|^2\}$$
$$\leqslant\int_T^t|f(s)|^2\,ds+c\int_T^t(\|z(s)\|^2+|z'(s)|^2)\,ds.$$

By Gronwall's inequality, for $T\leqslant t\leqslant T+\tau$,

$$\|z(t)\|^2+|z'(t)|^2\leqslant[\min(1,\alpha_0)]^{-1}\,e^{c(t-T)}\int_T^t|f(s)|^2\,ds.$$

The lemma follows immediately. A similar argument establishes

LEMMA 1.4. *Under the assumptions of Theorem* 1.1, *there is a positive function $c(t)$ such that*

$$\{\|u(t)\|^2+|u'(t)|^2\}^{1/2}\leqslant c(t)\{\|u_0\|^2+|u_1|^2+\|f\|^2_{L^2(0,T;H)}\}^{1/2}. \tag{1.6}$$

Moreover $c(t)$ is continuous and monotone increasing on $[0,+\infty)$ with $1\leqslant c(t)<+\infty$ for $0\leqslant t<+\infty$ and $\lim_{t\to+\infty}c(t)=+\infty$.

Let us provide $B(T)=\{u\in C([0,T];\ V): u'\in C([0,T];H)\}$ with the Banach space norm

$$\|u\|_{B(T)}=\max_{0\leqslant t\leqslant T}\{\|u(t)\|^2+|u'(t)|^2\}^{1/2}.$$

Let $D(T)=V\times H\times L^2(0,T;H)$ with the norm $\|(u_0,u_1,f)\|_{D(T)}=\{\|u_0\|^2+|u_1|^2+\|f\|^2_{L^2(0,T;H)}\}^{1/2}$. From Theorem 1.1 and Lemma 1.4 we have the following

COROLLARY 1.5. *There is a continuous linear map $K\colon D(T)\to B(T)$ with $K(u_0,u_1,f)=u$, where u is the unique solution of* (1.1) *and* (1.2). *In addition $\|u\|_{B(t)}\leqslant c(t)\|(u_0,u_1,f)\|_{D(T)}$, $0\leqslant t\leqslant T$, where $c(t)$ is the function from Lemma* 1.4.

2. Local Existence and Continuation

Let $R_0 > 0$ be a fixed number (possibly $R_0 = +\infty$). We henceforth assume that g is a nonlinear map from the open ball $S(R_0) = \{v \in V: \|v\| < R_0\}$ into H. Let r be any number, $0 < r < R_0$, and denote by $\bar{S}(r)$ the closed ball $\{v \in V: \|v\| \leqslant r\}$. We assume there exists a positive monotone nondecreasing function k defined on $[0, R_0)$ such that

$$|g(u) - g(v)| \leqslant k(r)\|u - v\|, \qquad \forall u, v \in \bar{S}(r). \tag{2.1}$$

This property of g as map from $\bar{S}(r)$ into H gives rise to the following time dependent property.

Lemma 2.1. *Let a, b, and r be real numbers with $a < b$ and $0 < r < R_0$. The map $g: S(R_0) \to H$ generates an operator from the set $C([a, b]; \bar{S}(r))$ into the space $L^2(a, b; H)$. Using g (for convenience) also to denote this operator, we have*

$$\| g(u) - g(v)\|_{L^2(a,b;H)} \leqslant (b-a)^{1/2}\, k(r)\|u - v\|_{C([a,b];V)} \tag{2.2}$$

for all u, $v \in C([a, b]; \bar{S}(r))$.

Proof. Since $t \to g(u(t))$ is a continuous map from $[a, b]$ into H we know that $g(u) \in L^2(a, b; H)$ whenever $u \in C([a, b]; \bar{S}(r))$. The estimate (2.2) follows easily from (2.1).

Let $0 < T < +\infty$. We define another map $\mathscr{F}$ from the set $C([0, T]; \bar{S}(r))$ into the space $B(T) \subset C([0, T]; V)$ by setting $\mathscr{F}(u) = K(u_0, u_1, g(u))$, where $u_0 \in V$, $u_1 \in H$ are given. As we pointed out in the previous lemma $g(u) \in L^2(0, T; H)$ whenever $u \in C([0, T]; \bar{S}(r))$; thus $\mathscr{F}$ is well-defined. If $w = \mathscr{F}(u)$ then w is the unique solution of (1.1) and (1.2) with $f(t) = g(u(t))$, $0 < t < T$, and w satisfies the energy equality (1.3). Therefore, if $u = \mathscr{F}(u)$ then u satisfies

$$u'' + A(t)\, u = g(u), \qquad 0 < t < T, \tag{2.3}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{2.4}$$

and also the energy equality

$$\begin{aligned} a(t; u(t), u(t)) &+ |u'(t)|^2 \\ &= a(0; u_0, u_0) + |u_1|^2 \\ &\quad + \int_0^t \{a'(s, u(s), u(s)) + 2(g(u(s)), u'(s))\}\, ds. \end{aligned} \tag{2.5}$$

Conversely, if u solves (2.3) and (2.4) then u satisfies (1.1) and (1.2) with

$f(t)=g(u(t))$, $0<t<T$. Hence by uniqueness $u=K(u_0,u_1,g(u))$ and u satisfies the energy equality (2.5) as well. Thus the fixed poins of $\mathscr{F}$ in $C([0,T];\bar{S}(r))$ are precisely the solutions of (2.3) and (2.4) satisfying $\|u(t)\|\leqslant r$, $0\leqslant t\leqslant T$, and each solution satisfies (2.5).

The following estimate will be useful in proving the subsequent theorems.

LEMMA 2.2. *For all* u, $v\in C([0,T];\bar{S}(r))$, *where* $T\in(0,+\infty)$ *and* $r\in(0,R_0)$, *we have*

$$\|\mathscr{F}(u)-\mathscr{F}(v)\|_{C([0,T];V)}\leqslant T^{1/2}c(T)\,k(r)\|u-v\|_{C([0,T];V)}, \qquad (2.6)$$

where $c(T)$ *denotes the function from Lemma* 1.4.

Proof. This follows immediately from (1.6) and (2.2) once we have written $\mathscr{F}(u)-\mathscr{F}(v)=K(0,0,g(u)-g(v))$.

THEOREM 2.3. (Local Existence and Uniqueness). *Let* $r_0\in(0,R_0)$. *If* $\|u_0\|\leqslant r_0$ *then there exists* $T>0$, T *depending on* u_0 *and* u_1, *such that the initial value problem* (2.3) *and* (2.4) *has a unique solution* $u\in B(T)$.

Proof. Let $\xi\in(0,1)$ and $r\in(r_0,R_0)$ be fixed. We take real numbers ε, $\delta>0$ so that $r_0+\varepsilon+2\delta\leqslant r$. Let us define a constant map f: $[0,+\infty]\to H$ by setting $f(t)=g(u_0)$ for each t. We note that $f\in L^2(0,T;H)$ for any $T\in(0,+\infty)$. Set $\omega=K(u_0,u_1,f)$. Since ω is continuous at 0, with $\omega(0)=u_0$, there exists $\tau>0$ such that $\|\omega(t)-u_0\|<\varepsilon$ for all $t\in(0,\tau)$. In general τ depends on u_0 and u_1. Notice that $\|\omega(t)\|\leqslant r_0+\varepsilon<r$ for $t\in[0,\tau]$. Choose $T>0$ so that $T\leqslant\tau$ and $T^{1/2}c(T)\leqslant\min\{\xi/k(r),\ \delta/((r+r_0)\,k(r))\}$. We define a closed subset of $C([0,T];V)$ by setting

$$E=\{u\in C([0,T];V): u(0)=u_0 \text{ and } \|u-\omega\|_{C([0,T];V)}\leqslant\delta\}.$$

Since $T\leqslant\tau$ we know $\omega\in C([0,T];\bar{S}(r))$; if $u\in E$ then for $0\leqslant t\leqslant T$ we have $\|u(t)\|\leqslant\|\omega(t)\|+\delta\leqslant r_0+\varepsilon+\delta<r$. Hence $E\subset C([0,T];\bar{S}(r))$. Moreover, letting u_0 denote the constant map $t\to u_0$ and using Lemma 2.2, we have for $u\in E$

$$\begin{aligned}\|\mathscr{F}(u)-\omega\|_{C([0,T];V)}&=\|\mathscr{F}(u)-\mathscr{F}(u_0)\|_{C([0,T];V)}\\&\leqslant T^{1/2}c(T)\,k(r)\|u-u_0\|_{C([0,T];V)}\\&\leqslant T^{1/2}c(T)\,k(r)(r+r_0)\leqslant\delta.\end{aligned}$$

Similarly, for $u,v\in E$

$$\|\mathscr{F}(u)-\mathscr{F}(v)\|_{C([0,T];V)}\leqslant\xi\,\|u-v\|_{C([0,T];V)}.$$

Thus $\mathscr{F}: E\to E$ is a contraction and consequently has a unique fixed point u in E. Since $u=\mathscr{F}(u)=K(u_0,u_1,g(u))$ it follows that $u\in B(T)$.

We show that u is unique. Suppose that $\tilde{u} \in C([0, T_0]; V)$ is another function satisfying

$$u'' + A(t)\,u = g(u), \qquad 0 < t < T_0 \leqslant T,$$
$$u(0) = u_0, \qquad u'(0) = u_1.$$

Since both $\tilde{u}$ and ω are continuous at 0, with $\tilde{u}(0) = u_0 = \omega(0)$, there exists $\tau > 0$ such that $\|\tilde{u} - \omega\|_{C([0,\tau];V)} \leqslant \delta$. By the fixed point argument above $\tilde{u} = u$ on $[0, \tau]$. Let $T_1 = \sup\{\tau \in (0, T_0): \tilde{u} = u \text{ on } [0, \tau]\}$, $T_1 \geqslant \tau > 0$. Clearly we have $\tilde{u}(T_1) = u(T_1)$. Suppose that $T_1 < T_0$. Then by continuity there exists $\tau > 0$, $T_1 + \tau \leqslant T_0$, such that $\|\tilde{u}(t) - u(t)\| < \delta$ for all $t \in [T_1, T_1 + \tau)$. Now $\tilde{u} = u$ on $[0, T_1]$ and so $\|u - \omega\|_{C([0,T_1];V)} \leqslant \delta$. For $t \in [T_1, T_1 + \tau]$ we have $\|\tilde{u}(t) - \omega(t)\| \leqslant 2\delta$ by the triangle inequality. Hence $\|\tilde{u} - \omega\|_{C([0,T_1+\tau];V)} \leqslant 2\delta$. Define $\tilde{E} = \{v \in C([0, T_1 + \tau]; V): v(0) = u_0$ and $\|v - \omega\|_{C([0,T_1+\tau];V)} \leqslant 2\delta\}$. We have $\tilde{E} \subset C([0, T_1 + \tau]; \bar{S}(r))$ and u, $\tilde{u} \in E$. Arguments similar to those above show that $\mathscr{F}: \tilde{E} \to \tilde{E}$ is a contraction and we conclude $u = \tilde{u}$ on $[0, T_1 + \tau]$. But this contradicts the definition of T_1; it must be that $T_1 = T_0$. Finally, if $T_0 < T$ then u is the unique extension of $\tilde{u}$ to a solution on $(0, T)$.

Remark. If g is defined on all of V (i.e., $R_0 = +\infty$) then we can take r_0 sufficiently large to show that local existence holds for arbitrary initial data.

In the next theorem we show that solutions can be continued uniquely to a larger interval, provided that $\|u(t)\|$ is bounded away from R_0 on $[0, T]$.

THEOREM 2.4 (Unique Continuation). *Suppose that u satisfies* (2.3) *and* (2.4) *and $\|u(t)\| \leqslant r_1 < R_0$ for all $t \in [0, T]$. Then there exists $\tau > 0$ and a unique extension of u to a solution of the inital value problem on the interval $[0, T + \tau]$.*

Proof. Let $\xi \in (0, 1)$ and $r \in (r_1, R_0)$ be fixed. Again we take numbers ε, $\delta > 0$ so that $r_1 + \varepsilon + 2\delta \leqslant r$. We define an extension $\bar{u}$ of u to the interval $[0, +\infty)$ by setting $\bar{u}(t) = u(t)$, $0 \leqslant t \leqslant T$, and $\bar{u}(t) = u(T)$, $T < t < +\infty$. Let $f(t) = g(\bar{u}(t))$ and observe that $f \in L^2(0, T_0; H)$ for all $T_0 \in (T, +\infty)$. We set $\omega = K(u_0, u_1, g(\bar{u}))$. By Lemma 1.2, $\omega|_{(0,T)} = K(u_0, u_1, g(u))$ on $(0, T)$ and hence by uniqueness $\omega = u$ on $[0, T]$. Furthermore since ω is continuous at T there exists $\eta > 0$ such that $\|\omega(t) - u(T)\| < \varepsilon$ for all $t \in [T, T + \eta)$. Choose $\tau > 0$ so that $\tau \leqslant \eta$ and $\tau^{1/2} c(\tau) \leqslant \min\{\delta/(r + r_1)\,k(r), \xi/k(r)\}$, where $c(\tau)$ denotes the function from Lemma 1.3. Since $\|\omega(t)\| \leqslant r_1 + \varepsilon < r$, for $0 \leqslant t \leqslant T + \tau$, we know $\omega \in C([0, T + \tau]; \bar{S}(r))$. Let us define

$$E = \{v \in C([0, T + \tau]; V): v = u \text{ on } [0, T] \text{ and } \max_{T \leqslant t \leqslant T + \tau} \|v(t) - \omega(t)\| \leqslant \delta\}.$$

If $v \in E$ then

$$\begin{aligned}\|v\|_{C([0,T+\tau];V)} &\leqslant \|\omega\|_{C([0,T+\tau];V)} + \delta \\ &\leqslant r_1 + \varepsilon + \delta < r.\end{aligned}$$

Hence $E \subset C([0, T+\tau]; \bar{S}(r))$. From Lemmas 1.2 and 1.3 it follows that

$$\begin{aligned}\|\mathscr{F}(v) - \omega\|_{C([0,T+\tau];V)} &= \|\mathscr{F}(v) - \mathscr{F}(\bar{u})\|_{C([0,T+\tau];V)} \\ &\leqslant \tau^{1/2} c(\tau)\, k(r)(r + r_1) \leqslant \delta;\end{aligned}$$

and for $v_1, v_2 \in E$

$$\begin{aligned}\|\mathscr{F}(v_1) - \mathscr{F}(v_2)\|_{C([0,T+\tau];V)} &\leqslant \tau^{1/2} c(\tau)\, k(r) \|v_1 - v_2\|_{C([0,T+\tau];V)} \\ &\leqslant \xi \|v_1 - v_2\|_{C([0,T+\tau];V)}.\end{aligned}$$

Thus $\mathscr{F}: E \to E$ is a contraction and has a unique fixed point in E. The argument for uniqueness of the extension is the same as in Theorem 2.3.

COROLLARY 2.5 (large time uniqueness). *For any $T > 0$ there can be at most one solution of* (2.3) *and* (2.4) *satisfying $\|u(t)\| \leqslant r_1 < R_0$ for some $r_1 \in (0, R_0)$ and all $t \in [0, T]$.*

Proof. Let u_1, u_2 be two solutions. By local uniqueness $u_1 = u_2$ on $[0, \tau]$ for some $\tau > 0$. Let $T_1 = \sup\{\tau \in (0, T): u_1 = u_2 \text{ on } [0, \tau]\}$ and set $u = u_1 = u_2$ on $[0, T_1]$. If $T_1 < T$ then u_1 and u_2 are two different extensions of u to a solution on $(0, T_2)$ for any $T_2 \in (T_1, T)$. By the previous theorem this is impossible. Hence $T_1 = T$ and $u_1 = u_2$ on $[0, T]$.

3. GLOBAL EXISTENCE

In this section we assume that the operator A is independent of t and coercive; that is, $a(u, u) \geqslant \alpha_0 \|u\|^2$ for each $u \in V$. In this case we can (and shall) use the equivalent norm $\|u\|_A^2 = a(u, u)$ on V. For convenience of notation we still write $\|\cdot\|$. We intend to show that global solutions (in time) exist for the initial value problem

$$u'' + Au = \varepsilon\phi(u), \qquad t > 0, \tag{3.1}$$

$$u(0) = u_0, \qquad u'(0) = u_1, \tag{3.2}$$

provided that ε is small. The function ϕ is to have the same local Lipschitz property, (2.1), assumed of g in the previous section. We further assume the existence of a functional, called a potential, $\Phi: S(R_0) \to \mathbb{R}$, satisfying $\Phi(0) = 0$ and

$$\frac{d}{dt}\Phi(u(t)) = (\phi(u), u'), \quad 0 < t < T, \forall u \in B(T) \cap C([0, T]; S(R_0)), \tag{3.3}$$

$$|\Phi(u)| \leqslant (k_0 + k_1(r)\|u\|)\|u\|, \quad \forall u \in \bar{S}(r), \tag{3.4}$$

where $k_0 \geqslant 0$ and $k_1(r)$ is a positive monotone nondecreasing function defined on $[0, R_0)$. As a consequence of these assumptions we have the following a priori estimate.

LEMMA 3.1. *We assume* $\{\|u_0\|^2 + |u_1|^2\}^{1/2} < r_1 < r < R_0$. *If* $u \in C([0, T]; \bar{S}(r))$ *is a solution of* (3.1) *and* (3.2) *on* $(0, T)$ *and*

$$0 < \varepsilon < \frac{r_1^2 - (\|u_0\|^2 + |u_1|^2)}{k_0^2 + 2|\Phi(u_0)| + r_1^2(1 + 2k_1(r))} \tag{3.5}$$

then $u \in C([0, T]; \bar{S}(r_1))$.

Proof. We use energy equality (2.5) and assumption (3.3) to write

$$\|u(t)\|^2 + |u'(t)|^2 = \|u_0\|^2 + |u_1|^2 + 2\varepsilon\{\Phi(u(t)) - \Phi(u_0)\}, \quad 0 \leqslant t \leqslant T.$$

Now by force of (3.4)

$$\begin{aligned}\|u(t)\|^2 &\leqslant \|u_0\|^2 + |u_1|^2 + 2\varepsilon\{(k_0 + k_1(r)\|u(t)\|)\|u(t)\| + |\Phi(u_0)|\} \\ &\leqslant \|u_0\|^2 + |u_1|^2 + \varepsilon\{k_0^2 + 2|\Phi(u_0)| + (1 + 2k_1(r))\|u(t)\|^2\}.\end{aligned}$$

Whence

$$\|u(t)\|^2 \leqslant \frac{\|u_0\|^2 + |u_1|^2 + \varepsilon(k_0^2 + 2|\Phi(u_0)|)}{1 - \varepsilon(1 + 2k_1(r))} \leqslant r_1^2, \qquad 0 \leqslant t \leqslant T,$$

the latter inequality being equivalent to (3.5). Notice that from (3.5) we have that $1 > \varepsilon(1 + 2k_1(r))$.

THEOREM 3.2 (global existence). *Let* $r_0 \in (0, R_0)$. *If* $\{\|u_0\|^2 + |u_1|^2\}^{1/2} \leqslant r_0 < r_1 < r < R_0$ *and* ε *satisfies* (3.5) *then there exists a unique global solution of* (3.1) *and* (3.2).

Proof. From the local existence theorem we know that a unique solution u^* exists on the interval $(0, T)$ for some $T > 0$ and $u^* \in C([0, T]; \bar{S}(r))$. We define a set $I = \{T \in (0, +\infty): u^*$ is a solution on $(0, T)$ with $u \in C([0, T]; S(r))\}$. Note that if $T \in I$ then $t \in I$ for all $t \in (0, T)$ by Lemma 1.2; hence I is a nonempty connected set. If $T \in I$ then by the previous Lemma $u \in C([0, T]; \bar{S}(r_1))$. Therefore by the continuation theorem, if $T \in I$ there exists $\tau > 0$ such that $T + \tau \in I$; hence I is open. We show that I is closed and therefore $I = (0, +\infty)$.

Let $\{T_n\}_{n=1}^{\infty}$ be a sequence in I which converges to T_0. If $T_0 = +\infty$ then $I = (0, +\infty)$ and there is nothing to prove. We therefore assume $T_0 < +\infty$. Also, we need only consider the case $T_n < T_0$ for every n. From Lemma 3.1 we have $u \in C([0, T_n]; \bar{S}(r_1))$ for each n and thus $u \in C([0, T_0); \bar{S}(r_1))$. Hence $g(u) \in L^2(0, T_0; H)$. Let $\omega = K(u_0, u_1, g(u))$ be the unique solution of the linear problem with $f(t) = g(u(t))$. We have $u(t) = \omega(t)$, $0 \leqslant t < T_0$, and therefore u extends to a solution on $(0, T_0)$. Moreover $u \in C([0, T_0]; \bar{S}(r_1))$ and thus $T_0 \in I$.

Remark. If ϕ is defined on all of V (i.e., $R_0 = +\infty$) then we can take r_1, r sufficiently large to show that (3.1) and (3.2) have a unique global solution for arbitrary initial data, provided that ε satisfies (3.5).

4. Global Nonexistence

We again take $g(u) = \varepsilon\phi(u)$, where $\varepsilon > 0$ and $\phi: S(R_0) \to H$ satisfies (2.1). Let A be independent of t and let $\lambda_1 \in \mathbb{R}$, $\omega_1 \in V$ be an eigenvalue, vector pair for A. Thus $a(\omega_1, v) = \lambda_1(\omega_1, v)$ for all $v \in V$.

Let $R_1 = c_0 |\omega_1| R_0$, where c_0 is the embedding constant for $V \subset H$. Then $\|u(t)\| < R_0$ implies $(u(t), \omega_1) < R_1$. Or, stating the countrapositive $(u(t), \omega_1) \geqslant R_1$ implies $\|u(t)\| \geqslant R_0$. Let $\mathscr{S}$ be the class of functions $\psi \in C(-\infty, R_1)$ with the property that $\psi(s) \geqslant y_0 > 0$ for $s_0 \leqslant s < R_1$, for a pair of fixed numbers s_0, y_0. We further assume of ϕ that for some $\psi \in \mathscr{S}$

$$(\phi(u), \omega_1) \geqslant \psi((u, \omega_1)) \qquad \forall u \in S(R_0). \tag{4.1}$$

This can be thought of as an abstract Jensen's inequality as we shall see in the application. To this function ψ we associate three functions which arise naturally in the proof of Theorem 4.2. Let $\Psi \in C^1(-\infty, R_1)$ be the unique antiderivative of ψ satisfying $\Psi(0) = 0$. For $c, s \in (-\infty, R_1)$ and $\varepsilon > 0$ we set

$$H(s, \varepsilon) = -\tfrac{1}{2}\lambda_1 s^2 + \varepsilon\Psi(s),$$

and

$$\omega(s, c, \varepsilon) = 2[H(s, \varepsilon) - H(c, \varepsilon)].$$

The following lemma will be used in proving Theorem 4.2. We assume henceforth that ψ is the function in (4.1) with s_0, y_0 fixed.

Lemma 4.1. *Let* $\varepsilon_1 = \lambda_1 \sup\{s[\psi(s)]^{-1}: s_0 \leqslant s < R_1\}$. *For all* $\varepsilon > \varepsilon_1$

(i) *$H(s, \varepsilon)$ is strictly increasing in s for $s_0 \leqslant s < R_1$,*

(ii) *$\omega(s, c, \varepsilon)$ is strictly increasing in s for $s_0 \leqslant s < R_1$ and therefore $\omega(s, c, \varepsilon) > 0$ for $s > c$,*

(iii) *for $s_0 \leqslant c < R_1$ we have*

$$\int_c^{R_1} [\omega(s, c, \varepsilon)]^{-1/2}\, ds < +\infty.$$

Proof. Taking a partial derivative we find for $s_0 \leqslant s < R_1$

$$H_s(s, \varepsilon) = -\lambda_1 s + \varepsilon\psi(s) = \psi(s)\left[\varepsilon - \lambda_1 \frac{s}{\psi(s)}\right] > 0.$$

Thus (i) and (ii) follow immediately. Notice that

$$\lim_{s \to c} \frac{\omega(s, c, \varepsilon)}{s - c} = 2H_s(c, \varepsilon) > 0, \qquad s_0 \leqslant c < R_1.$$

Hence there exists $\delta > 0$ such that $\omega(s, c, \varepsilon) > (s - c)\, H_s(c, \varepsilon)$ for $c < s < c + \delta$; and

$$\int_c^{c+\delta} [\omega(s, c, \varepsilon)]^{-1/2}\, ds < +\infty.$$

Since $\omega(s, c, \varepsilon) \geqslant \omega(c + \delta, c, \varepsilon) > 0$ for $s \geqslant c + \delta$ we have established (iii).

We now consider nonexistence of global solutions. Let u be a solution on $(0, T)$ of

$$u'' + Au = \varepsilon\phi(u), \qquad 0 < t < T, \tag{4.2}$$

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{4.3}$$

Of course $u(t) \in$ domain $\phi = S(R_0)$ (a.e.) $t \in (0, T)$. The next theorem shows that we cannot have solutions of (4.2) and (4.3) for all $T > 0$ if ε is sufficiently large. In the proof we use the functions $H(s) = H(s, \varepsilon)$ and $\omega(s, c) = \omega(s, c, \varepsilon)$ defined above with the dependence on ε suppressed.

THEOREM 4.2. *Let $u_0 \in V$, $u_1 \in H$ be given with $\|u_0\| < R_0$. Let ϕ satisfy (4.1) with $\psi(s) \geqslant y_0 > 0$ for $s_0 \leqslant s < R_1$ and let $\varepsilon > \varepsilon_1 = \lambda_1 \sup\{s[\psi(s)]^{-1}: s_0 \leqslant s < R_1\}$. If $(u_0, \omega_1) \geqslant s_0$ and $(u_1, \omega_1) \geqslant 0$ then the local solution $u(t)$ cannot be continued to a global solution.*

Proof. Suppose that $u(t)$ can be continued to a global solution; then we must have $\|u(t)\| < R_0$ for almost all $t > 0$. Let $U(t) = (u(t), \omega_1)$ and observe that $|U(t)| \leqslant c_0 \|u(t)\| |\omega_1| < R_1$ (a.e.) $t > 0$. Since $u(t)$ is a solution of (4.2) and (4.3) and $\omega_1 \in V$ it follows from the definition of weak solution that $U'' + \lambda_1 U = \varepsilon(\phi(u), \omega_1)$ for $t > 0$. Hence $U'' + \lambda_1 U \geqslant \varepsilon\psi(U)$ or $U'' \geqslant -\lambda_1 U + \varepsilon\psi(U) = H'(U)$ for all $t > 0$. Since $U''(0) \geqslant H'(U(0)) = H'((u_0, \omega_1)) > 0$ by the previous lemma and $U'(0) \geqslant 0$ by assumption it follows that for some $\eta > 0$, $U'(t) > 0$ on $(0, \eta)$. Thus

$$U''(t)\, U'(t) \geqslant H'(U(t))\, U'(t), \qquad 0 < t < \eta.$$

After a quadrature we find

$$U'(t) \geqslant \{|U'(0)|^2 + \omega(U(t), U(0))\}^{1/2}, \qquad 0 < t < \eta.$$

From this we deduce that $U'(t) > 0$ for all $t > 0$, owing to the fact that $\omega(s, U(0))$ is positive for all $s \in (U(0), R_0)$. Therefore we have, for all $t > 0$,

$$\begin{aligned} t &\leqslant \int_0^t \{|U'(0)|^2 + \omega(U(t), U(0))\}^{-1/2}\, U'(t)\, dt \\ &= \int_{U(0)}^{U(t)} \{|U'(0)|^2 + \omega(s, U(0))\}^{-1/2}\, ds \\ &\leqslant \int_{U(0)}^{R_1} \{\omega(s, U(0))\}^{-1/2}\, ds. \end{aligned}$$

Since the last integral is finite by Lemma 4.1, this is clearly a contradiction. Therefore u cannot be continued to a global solution.

Remark. There may be a smaller value $\varepsilon_2 < \varepsilon_1$ for which we still have nonexistence of global solutions when $\varepsilon > \varepsilon_2$. Indeed what is actually needed in the proof are the implications that $\omega(s, U(0), \varepsilon) > 0$ for $U(0) < s < R_1$ and that the last integral above converges, whenever $\varepsilon > \varepsilon_2$. This does not necessarily require that $H(s, \varepsilon)$ be increasing on $s_0 \leqslant s < R_1$, as is guaranteed by taking $\varepsilon > \varepsilon_1$.

The proof of Theorem 4.2 actually shows that a necessary condition for the continuation of the local solution to a solution on the interval $(0, T)$ is that

$$T \leqslant \int_{U(0)}^{U(T)} \{|U'(0)|^2 + \omega(s, U(0))\}^{-1/2}\, ds.$$

Since $U(t) < R_1$ while $u(t) \in \operatorname{dom}(\phi)$, an upper bound for T is

$$T_0 = \int_{U(0)}^{R_1} \{|U'(0)|^2 + \omega(s, U(0))\}^{-1/2}\, ds.$$

If u can be continued to a solution on the interval $(0, T_0)$ then necessarily $U(T_0) = R_1$ and hence $\|u(T_0)\| = R_0$. That is u wanders out of the domain of ϕ at time T_0. We next show that this is essentially what happens for some time $T^* \leqslant T_0$.

THEOREM 4.3 (abstract quenching). *Under the hypothesis of Theorem* 4.2, *there exists* $T^* \in (0, T_0]$ *and a nondecreasing sequence* $\{t_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty} t_n = T^*$ *such that* u *is a solution on* $(0, t_n)$ *for every* n *and* $\lim_{n\to\infty} \|u(t_n)\| = R_0$.

Proof. We know there is a unique local solution u^* on $(0, T)$ for some $T > 0$ and of course $u^*(t) \in S(R_0)$, $0 \leqslant t \leqslant T$. We define a set

$$I = \{T \in (0, +\infty): u^* \text{ is a solution on } (0, T) \text{ and } u^* \in C([0, T]; S(R_0))\}.$$

Note that if $T \in I$ then $t \in I$ for all $t \in (0, T)$; hence I is nonempty and connected. Furthermore if $T \in I$ then $r = \max_{0 \leqslant t \leqslant T} \|u(t)\| < R_0$. Thus by continuation there exists $\tau > 0$ such that $T + \tau \in I$; hence I is open. Since $I \neq (0, +\infty)$ by the previous theorem, I is not closed. We must have $I = (0, T^*)$ for some $T^* \leqslant T_0$. Note that $T^* \notin I$. Let $\{T_n\}_{n=1}^{\infty}$ be an increasing sequence in I with $\lim_{n\to\infty} T_n = T^*$. We set $r_n = \max_{0 \leqslant t \leqslant T_n} \|u(t)\|$ and observe that $r_n < R_0$, $r_n \leqslant r_{n+1}$ for all n. Hence $r^* = \lim_{n\to\infty} r_n$ exists and $r^* \leqslant R_0$. Now if $r^* < R_0$ then by the same argument used in Theorem 3.2 one shows $T^* \in I$, a contradiction. Therefore we must have $r^* = R_0$. If t_n is selected so that $t_n \geqslant t_{n-1}$ and $\|u(t_n)\| = \max_{0 \leqslant t \leqslant T_n} \|u(t)\| = r_n$ then $t_n \to T^*$ and $\lim_{n\to\infty} \|u(t_n)\| = R_0$.

5. AN APPLICATION

We consider the problem of the nonlinear vibrating supported plate

$$u_{tt} + \Delta^2 u = \varepsilon\phi(u), \qquad (t, x) \in \mathbb{R}^+ \times \Omega, \tag{5.1}$$

$$u(t, x) = \Delta u(t, x) = 0, \qquad (t, x) \in \mathbb{R}^+ \times \partial\Omega, \tag{5.2}$$

$$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x) \qquad x \in \Omega, \tag{5.3}$$

where Ω is the rectangle $(0, L) \times (0, 1) \subset \mathbb{R}^2$, $0 < L \leqslant 1$, and $\phi \in C^1(-\infty, M)$ is convex with $\lim_{u\to M^-} \phi(u) = +\infty$. Here $M > 0$ and for simplicity we assume $\phi(0) > 0$, $\phi'(0) \geqslant 0$. In the previous setting we have $H = L^2(\Omega)$ and $V = H_0^1(\Omega) \cap H^2(\Omega)$. Alternately, V is the closure in $H^2(\Omega)$ of the set $D = \{\phi \in C^\infty(\bar{\Omega}): \phi(x) = \Delta\phi(x) = 0, x \in \partial\Omega\}$. We have $A = \Delta^2$ and

$$a(u, v) = \int_\Omega \Delta u \, \Delta v \, dx, \qquad u, v \in V.$$

Notice that $\|u\|^2 = a(u, u)$ defines a norm on V; for if $\|u\| = 0$ then $\Delta u = 0$ in Ω and $u = 0$ on $\partial\Omega$; hence $u = 0$. From the Sobolev embedding theorem we know $V \subset C(\bar{\Omega})$. If $(x_1, x_2) \in \Omega$ then

$$u(x_1, x_2) = \int_0^{x_2} u_s(x_1, s)\, ds = \int_0^{x_2} \int_0^{x_1} u_{sr}(r, s)\, dr\, ds.$$

Similarly

$$\begin{aligned} u(x_1, x_2) &= \int_{x_2}^{1} \int_{x_1}^{L} u_{sr}(r, s)\, dr\, ds \\ &= -\int_0^{x_2} \int_{x_1}^{L} u_{sr}(r, s)\, dr\, ds = -\int_{x_2}^{1} \int_0^{x_1} u_{sr}(r, s)\, dr\, ds. \end{aligned}$$

Thus we obtain

$$4\,|u(x_1, x_2)| \leqslant \int_0^1 \int_0^L |u_{sr}(r, s)|\, dr\, ds \leqslant L^{1/2}\, \|u_{x_1x_2}\|_{L^2(\Omega)}.$$

Hence $\|u\|_{L^\infty(\Omega)} \leqslant (L/16)^{1/2}\, \|u_{x_1x_2}\|_{L^2(\Omega)}$. Now for all $\phi \in D$ we have $\|\phi_{x_1x_2}\|_{L^2(\Omega)} \leqslant 2^{-1/2}\, \|\phi\|$ and passing to the limit shows the same is true for $u \in V$. This yields the embedding inequality.

$$\|u\|_{L^\infty(\Omega)} \leqslant (L/32)^{1/2}\, \|u\|, \qquad u \in V. \tag{5.4}$$

From the Fourier series representations of u, Δu and Parseval's identity we deduce the embedding inequality

$$|u| = \|u\|_{L^2(\Omega)} \leqslant \frac{L^2}{\pi^2(1 + L^2)}\, \|\Delta u\|_{L^2(\Omega)} = c_0\, \|u\|, \qquad u \in V. \tag{5.5}$$

We take the domain of ϕ to be the open ball $S(R_0)$ where $R_0 = M(32/L)^{1/2}$; if $\|u\| < R_0$ then $|u(x)| < M$ for all $x \in \Omega$. For any $r \in (0, R_0)$ we set $K(r) = \max\{|\phi'(z)| : |z| \leqslant r(L/32)^{1/2}\}$. If u, $v \in \bar{S}(r) = \{u \in V: \|u\| \leqslant r\}$ then

$$\begin{aligned} |\phi(u) - \phi(v)| &= \left\{\int_\Omega |\phi(u) - \phi(v)|^2\, dx\right\}^{1/2} \\ &\leqslant K(r) \left\{\int_\Omega |u - v|^2\, dx\right\}^{1/2} \leqslant k_0(r) \|u - v\|, \end{aligned}$$

where $k_0(r) = c_0 K(r)$. We define the functional Φ: $S(R_0) \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega \int_0^{u(x)} \phi(s)\, ds\, dx.$$

Clearly $\Phi(0)=0$. It is not difficult to show that

$$\frac{d}{dt}\Phi(u(t))=\int_\Omega \phi(u(t,x))\,u_t(t,x)\,dx=(\phi(u),u'), \qquad 0<t<T,$$

for $u\in\{v\in C([0,T];S(R_0)): v'\in C([0,T];H)\}$. By using Taylor's theorem we estimate for $u\in\bar{S}(r)$,

$$|\Phi(u)|\leqslant\int_\Omega \{|\phi(0)|+\tfrac{1}{2}K(r)|u(s)|\}|u(s)|\,dx.$$

By the Hölder and Minkowski inequalities we have

$$\begin{aligned}|\Phi(u)|&\leqslant\left\{\int_\Omega [|\phi(0)|+\tfrac{1}{2}K(r)|u(x)|]^2\,dx\right\}^{1/2}\left\{\int_\Omega |u(x)|^2\,dx\right\}^{1/2}\\&\leqslant(L^{1/2}\,|\phi(0)|+\tfrac{1}{2}K(r)\,|u|)\,|u|\\&\leqslant(k_0+k_1(r)\|u\|)\|u\|,\end{aligned}$$

where $k_0=c_0L^{1/2}\,|\phi(0)|$ and $k_1(r)=\frac{1}{2}c_0^2K(r)$.

An eigenvalue-vector pair for Δ^2 (plus the boundary conditions) is $\lambda_1=\pi^4(1+L^{-2})^2$ and $\omega_1(x)=(\pi^2/4L)\sin(\pi x_1/L)\sin\pi x_2$. If $u\in S(R_0)$ then $|u(x)|<M$ for all $x\in\Omega$. Thus we may apply Jensen's inequality to show

$$\begin{aligned}(\phi(u),\omega_1)&=\int_\Omega \phi(u(x))\,\omega_1(x)\,dx\\&\geqslant\phi\left(\int_\Omega u(x)\,\omega_1(x)\,dx\right)=\phi((u,\omega_1)).\end{aligned}$$

We set $R_1=c_0|\omega_1|R_0=ML/\sqrt{2}\,(1+L^2)$ and observe that $R_1<M$; hence $\phi\in c^1(-\infty,R_1)$. Thus we can take $\psi=\phi$ in (4.1) with $s_0=0$, $y_0=\phi(0)$ so that $\psi(s)\geqslant y_0$ if $s\geqslant s_0$.

Having verified all the needed hypotheses we now state Theorem 5.1 below as a corollary to the previous general results. For convenience we set

$$N_0=\left\{\int_\Omega |\Delta u_0|^2\,dx\right\}^{1/2}, \qquad N_1=\left\{\int_\Omega |u_1|^2\,dx\right\}^{1/2}.$$

THEOREM 5.1. (a) *If* $(N_0^2+N_1^2)^{1/2}\leqslant r_0<M(32/L)^{1/2}$, r_1 *is any number such that* $r_0<r_1<M(32/L)^{1/2}$, *and* ε_0 *is defined by*

$$\varepsilon_0=\frac{r_1^2-(N_0^2+N_1^2)^{1/2}}{k_0^2+2\,|\Phi(u_0)|+r_1^2(1+2k_1(r))},$$

then global solutions of (5.1)–(5.3) *exist for all* ε, $0<\varepsilon<\varepsilon_0$.

(b) *If* $N_0 < M(32/L)^{1/2}$,

$$\int_\Omega u_0(x) \sin \frac{\pi x_1}{L} \sin \pi x_2 \, dx \geqslant 0,$$

$$\int_\Omega u_1(x) \sin \frac{\pi x_1}{L} \sin \pi x_2 \, dx \geqslant 0,$$

and $\varepsilon > \varepsilon_1 = \pi^4(1 + L^{-2})^2 \sup\{s[\phi(s)]^{-1}: 0 \leqslant s < R_1\}$, *then* (5.1)–(5.3) *have a unique local solution which cannot be continued to a global solution in time.*

In this case we can prove more than nonexistence of global solutions. We can actually show that the solutions must quench in finite time, in the sense that $m(t) = \max\{u(t, x): x \in \Omega\}$ reaches M at some time $T \in (0, +\infty)$. If u is a solution on $[0, T]$ and $m(T) < M$ then we claim that u can be extended to a solution on a larger interval. Since $m(t) < M$ does not necessarily imply that $\|u(t)\|_V < R_0$ we need to modify the arguments of Section 2 to obtain this continuation result. Notice that we only have knowledge of $u(t)$ in the weaker topology of $L^\infty(\Omega)$. We therefore replace the space V by the space $L^\infty(\Omega)$.

Let $S_\infty(M) = \{u \in L^\infty(\Omega): \|u\|_{L^\infty(\Omega)} < M\}$. Clearly for $\phi \in C^1(-\infty, M)$ we have $\phi: S_\infty(M) \to H = L^2(\Omega)$. For any $r \in (0, M)$ we set $\bar{S}_\infty(r) = \{u \in L^\infty(\Omega): \|u\|_{L^\infty(\Omega)} \leqslant r\}$. If $k(r) = L^{1/2}\max\{|\phi'(s)|: |s| \leqslant r\}$ then $|\phi(u) - \phi(v)| \leqslant k(r)\|u - v\|_{L^\infty(\Omega)}$ for all u, $v \in \bar{S}_\infty(r)$. Thus ϕ has property (2.1) in the weaker topology of $L^\infty(\Omega)$. If $u \in C([a, b]; \bar{S}_\infty(r))$ then $\phi(u) \in L^2(a, b; H)$ and

$$\|\phi(u) - \phi(v)\|_{L^2(a,b;H)} \leqslant (b - a)^{1/2} k(r)\|u - v\|_{C([a,b];L^\infty(\Omega))}.$$

Hence ϕ generates an operator from $C([a, b]; \bar{S}_\infty(r))$ into $L^2(a, b; H)$ which has the local Lipschitz property above.

Let $0 < T < +\infty$ and define $\mathscr{F}_\infty: C([0, T]; \bar{S}_\infty(r)) \to C([0, T]; L^\infty(\Omega))$ by setting $\mathscr{F}_\infty(u) = i \circ K(u_0, u_1, \phi(u))$ where i denotes the inclusion map $B(T) \subset C([0, T]; L^\infty(\Omega))$; recall that $V \subset L^\infty(\Omega)$. Observe that

$$\begin{aligned}
&\|\mathscr{F}_\infty(u) - \mathscr{F}_\infty(v)\|_{C([0,T];L^\infty(\Omega))} \\
&\quad = \|i \circ K(0, 0, \phi(u) - \phi(v))\|_{C([0,T];L^\infty(\Omega))} \\
&\quad \leqslant (L/32)^{1/2} \|K(0, 0, \phi(u) - \phi(v)\|_{C([0,T];V)} \\
&\quad \leqslant (L/32)^{1/2} c(T)\|\phi(u) - \phi(v)\|_{L^2(0,T;H)} \\
&\quad \leqslant (L/32)^{1/2} c(T) T^{1/2}k(r)\|u - v\|_{C([0,T];L^\infty(\Omega))}.
\end{aligned}$$

Having established these prerequisite estimates, it is now clear that the arguments of Theorems 2.3 and 2.4 can be repeated with V replaced throughout by $L^\infty(\Omega)$. Thus if $\|u(t)\|_{L^\infty(\Omega)} < M$ for all $t \in [0, T]$ then u can be extended to a larger interval $[0, T+\tau]$ for some $\tau > 0$.

We now show that under the hypotheses of Theorem 5.1(b) there is a unique local solution which quenches in finite time. In fact we already know that there is a unique local solution which cannot be continued to a global solution. From the proof of Theorem 4.2 we know that $U(t)$ is increasing so that $0 \leqslant U(0) \leqslant U(t)$. Since $\omega_1(x)\,dx$ is a probability measure we have $U(t) \leqslant m(t)$ and hence $m(t) \geqslant 0$ for all t in the interval of existence. If u is a solution on $[0, T]$ and $m(t) < M$, $0 \leqslant t \leqslant T$, then $\|u(t)\|_{L^\infty(\Omega)} = m(t) < M$, $0 \leqslant t \leqslant T$. Therefore u can be continued as a solution to a larger interval. But at any time t for which $m(t) < M$ we may apply Jensen's inequality to show $(\phi(u), \omega_1) \geqslant \phi((u, \omega_1))$. That is we may take $\psi = \phi$ in (4.1) with $\Psi(s)$, $H(s)$, and $\omega(s, c)$ now being defined for $-\infty < s < M$ and $-\infty < c < M$. Repeating the arguments in Theorem 4.2 shows that

$$t \leqslant \int_{U(0)}^{m(t)} \{|U'(0)|^2 + \omega(s, U(0))\}^{-1/2}\, ds < +\infty$$

for any t in the interval of existence. Clearly we must have $m(\bar{t}) = M$ at some time $\bar{t} \leqslant T_0$, where

$$T_0 = \int_{U(0)}^{M} \{|U'(0)|^2 + \omega(s, U(0))\}^{-1/2}\, ds < +\infty.$$

Remarks. (1) The above results show that for the problem

$$\begin{aligned} &u_{tt} + \Delta^2 u = \phi(u) && (t, x) \in \mathbb{R}^+ \times \Omega, \\ &u(t, x) = \Delta u(t, x) = 0 && (t, x) \in \mathbb{R}^+ \times \partial\Omega, \\ &u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x), && x \in \Omega, \end{aligned}$$

where $\Omega = (0, a) \times (0, b)$, $0 < a \leqslant b$, global solutions exist if b is small but do not exist if b is large. We change variables $\tilde{x}_1 = b^{-1}x_1$, $\tilde{x}_2 = b^{-1}x_2$, $\tilde{t} = b^{-2}t$ (a congruent scaling of the rectangle) to obtain (5.1)–(5.3) with $L = a/b$ and $\varepsilon = b^4$.

(2) We can consider more general domains Ω in (5.1)–(5.3). In fact Ω could be any reasonably smooth domain in $\mathbb{R}^2$ or $\mathbb{R}^3$; for in this case the first eigenfunction of the Laplacian (with $u = 0$ on $\partial\Omega$) is positive in Ω (cf. Courant and Hilbert [3]) and hence the biharmonic operator (with $u = \Delta u = 0$ on $\partial\Omega$) has a positive eigenfunction in Ω.

(3) The above methods also apply to the case in which Ω is a disc in the plane and the boundary conditions $u = \Delta u = 0$ on Ω are replaced by the boundary conditions $u = \partial u/\partial n = 0$ on Ω. In this case we again have a positive eigenfunction which can be used as ω_1.

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